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Longevity, Capital Formation and Economic Development

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Longevity, Capital Formation and Economic Development

Qiong Zhang

Abstract:

Many researchers have concluded that longer life expectancies prompt increased investment in education, as a prolonged labor supply raises the rate of return on education. Besides explaining the empirical evidence behind this conclusion (at an absolute level), there is another issue to be discussed: does time spent in studying and working increase proportionally with higher longevity? Building on an extended life-cycle model with an assumption on a more realistic distribution of life cycle mortality rates, this article considers dynamic effects of prolonging longevity on economic development by directly introducing changes in longevity into the economy, which is more preferable than comparative static analysis that relies on changes in relevant parameters. It shows that prolonged life expectancy will cause individuals to increase their time in education but may not warrant rises in labor input. Later we show that higher improvement rate of longevity will also promote economic growth, even we exclude the mechanism of human capital formation, and only consider growth effects of higher improvement rate of life expectancy from physical capital investment.

JEL codes: B10, D11, D91, E21, E24

Key words: longevity, education, labor supply, economic development

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1 Introduction

Since there is consensus that human capital formation is important for economic development, the relationship between increased longevity and investment in human capital has been studied extensively. Much of the relevant literature suggests that prolonged longevity can spur investment in human capital—and thus spur economic growth, due to the higher (expected) returns on a prolonged labor supply. This theoretical explanation was first formulated by Ben-Porath (1967), who concluded that a rise in life expectancy would be associated with a rise in lifetime labor input, and that both together would bring about a rise in investment in human capital and thus prompt economic growth. A number of subsequent studies follow this explanation: for example, Ehrlich and Lui (1991); de la Croix and Licandro (1999); Hall and Jones (1999); Bils and Klenow (2000); Kalemli-Ozcan, Ryder, and Weil (2000); Boucekkine, de la Croix, and Licandro (2002); Cervellati and Sunde (2005); Boldrin, Jones, and Khan (2005); Soares (2006); and Tamura (2006), among others. Many of these authors go so far as to argue that falling mortality rates (or gains in longevity) can induce individuals to make a quality-quantity trade-off and thus reduce fertility rates; this demographic transition would promote even higher growth rates and could explain how the transition from Malthusian stagnation to a state of sustained economic growth is triggered.[1] Other studies, however, present a somewhat different picture. Zhang, Zhang, and Lee (2001) show that longevity will increase human capital investment even if it does not directly affect the rate of return on such investment. Hazan and Zoabi (2006) argue that prolonging longevity does not necessarily lead to a rise in human capital formation. They conclude that a

[1] See Soares (2006). Refer to Galor (2005) for a more comprehensive historical survey of income per capita and human capital formation, along with explanations of their dynamics.

rise in the life expectancy of children increases not only returns on quality but also returns on quantity, mitigating the incentive of parents to invest more in their children's education.

Empirical studies, however, are less unanimous in their conclusions. For example, Acemoglu and Johnson (2007) use cross-country regressions and find no systematic relationship between life expectancy and economic growth. Lorentzen, McMillan, and Wacziarg (2008), Aghion, Howitt, and Murtin (2010) and Zhang (2010) find that both higher initial life expectancies and steeper increases in longevity have significant positive effects on income growth. The first two studies in particular emphasize that reductions in adult mortality rates, which influence labor input most, go far toward explaining the health effects of growth. Bilos and Klenow (2000) and Manuelli and Seshodri (2005) find a positive correlation, although of different magnitudes. In addition, Hazan (2009) observes that higher returns on education have to be realized by way of a longer working life. He therefore hypothesizes that higher life expectancy leads to greater investment in education, accompanied by an increase in lifetime labor supply. However, his time-series study of American and European men (1840–1970) finds no relation between life expectancy and lifetime labor supply.

Economic analyses of the effects of demographic changes, beginning with those of Yaari (1965) and Blanchard (1985), have typically used either age-specific mortality rates—the infant mortality rate (IMR), under-five mortality rate (U5MR), and maternity mortality rate (MMR) are the most common—or aggregate measurements such as life expectancy (LE); they sacrifice the details of age-specific mortality to make analytical progress. This simplification, however, may obscure the effects of demographic changes on economic development (see Li and Tuljapurker, 2004). In other words, considering only a rise in life expectancy (or a change in the mortality rates of specific age groups) is not enough to reveal the relationship between improved health and economic development (Sheshinsky, 2009).

The motivation for this article is to reveal the relationships among longevity and human capital investment and labor input, and to explain gaps among existing hypotheses and between these hypotheses and empirical evidence. Using a life-cycle model extended over infinite generations, it is demonstrated that prolonged longevity prompts increased investment in education but does not necessarily increase labor input. Instead, labor input depends on how life expectancies rise due to changes in lifetime survival probabilities over generations. This article uses a Weibull (1951) distribution as a representation of age at death. It demonstrates that an increase in life expectancy may lead to different decisions on investment in education and lifelong labor input; therefore, dynamic consumption, accumulation of physical investment, and economic development depend on changes in the hazard rates (or age-specific mortality rates which are commonly used in the field of health economics) of different age groups within the same generation and on changes in the hazard rates of the same age group over several generations.

This article combines methods and insights from Li and Tuljapurker (2004) and Sheshinsky (2009). That said, it diverges from Li and Tuljapurker (2004) by using a Weibull distribution, which is more consistent with realistic mortality patterns than a normal distribution. It also differs from these two studies as it considers dynamic physical accumulation across generations, therefore it considers human capital investment and labor input for each generation, and wealth reallocations among different generations. The salient feature of this article is that we use a more realistic distribution to represent overtime changes in life cycle survival probabilities and directly consider the effects of changes in longevity on economic growth, rather than only considering changes of mortality rates of some specific ages as commonly used in literature.

The remainder of this article is structured as follows. Section 2 outlines the relevant socioeconomic statistics of several industrialized countries over centuries between 1820 and 1992 and summarizes historical evidence to be explained in subsequent sections. Section 3 introduces the Weibull distribution in the context of survival and mortality, along with its associated survival function, life expectancy, and hazard rate. It relates the changes in hazard rates to changes in life expectancy, and parsimoniously transforms changes in hazard rates into changes in the parameter of the Weibull distribution. Section 4 builds an extended lifecycle model in order to effectively determine the effects of increased longevity on consumption, educational attainment, labor supply and productivity, and economic development. Section 5 concludes.

2 Economic Developments and Lifetime Survival Patterns over Centuries

Historical evidence demonstrates that human capital investment and longevity are positively correlated:[1] for example, the average number of years of schooling in England and Wales rose from 2.3 for the cohort born between 1801 and 1805, to 5.2 for the cohort born in the

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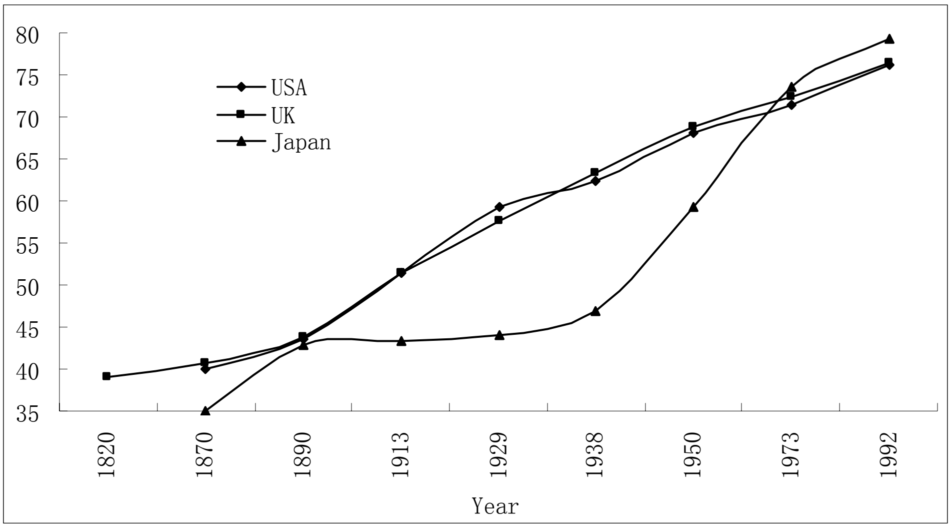
[1] Two definitions of life expectancy need to be clearly emphasized here: First, the period life expectancy at birth (PLE_0) for a given year represents the average number of years of life remaining if a group of persons were to experience *the mortality rates for that year* over the course of their remaining lives. Second, the cohort life expectancy at birth (CLE_0) for a given year represents the average number of years of life remaining if a group of persons were to experience *the mortality rates for the series of years* in which they reach each succeeding age. Though historical data show that these two life expectancies are very similar, the former tends to be more insensitive than the latter; i.e., PLE_0 will increase (decrease) less slowly when health improves (worsens), as PLE_0 includes effects on preceding generations. In this article, we refer to life expectancy as the period one unless specified otherwise.

years 1852–56; meanwhile, between 1740 and 1840 life expectancy at birth rose from 33 to 40 in England (Galor, 2005). This relationship is confirmed when we look at more recent data on select countries for which good-quality data are available.

As shown in Figure 1, life expectancies (LEs) at birth for the United Kingdom (in 1820),[1] the United States (in 1850),[2] and Japan (in 1850)[3] are 39, 39.4, and 35, respectively. As the populations of the United Kingdom and the United States maintained steady gains in health, their LEs respectively grew to 43.8 (in 1891)[4] and 42.5 (in 1890),[5] and then to 76.46 and 76.08 in 1992.[6] Meanwhile there was nearly no increase in life expectancy in Japan until after 1938. Since 1940, however, Japan has seen one of the highest rates of improved longevity; life expectancy at birth exceeded 79 in 1992, one of the highest figures in the world.

Figure 1: Life Expectancy at Birth in Select Countries (1820–1992)

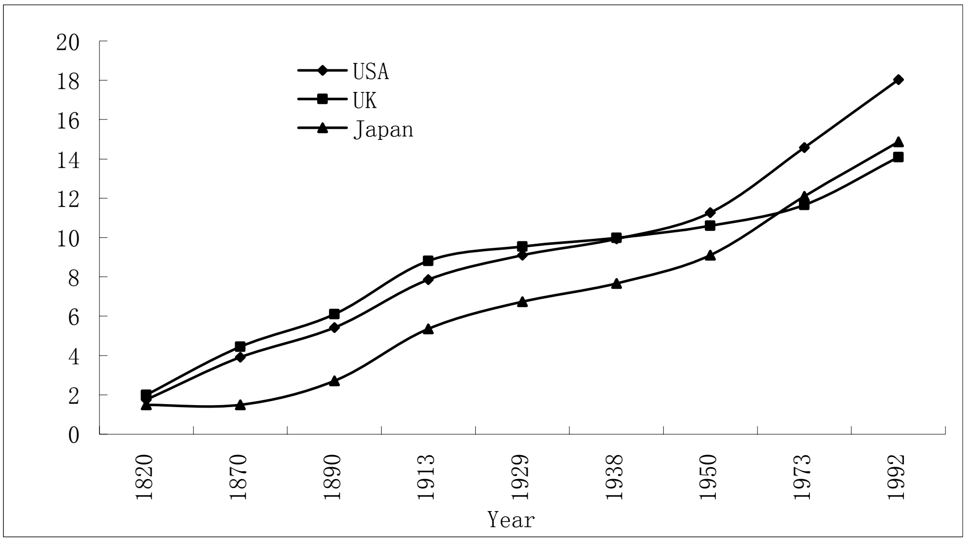


[1] See Maddison (1995).
[2] Data cover the average of female (40.5) and male (38.3) white and nonwhite residents of Massachusetts. Source: U.S. Dept. of Commerce, Bureau of the Census, Historical Statistics of the United States. Refer to the website http://www.infoplease.com/ipa/A0005140.html for more details.
[3] See Maddison (1995).
[4] Average female (45.71) and male (41.89) LEs of England and Wales; data are from Preston, Keyfitz, and Schoen (1972).
[5] Same as Footnote 4.
[6] The data source and calculating method for the United Kingdom (1911) are the same as given in Footnote 4 for the United States (period of 1909–11); data are from http://www.cdc.gov/nchs/data/nvsr/nvsr56/nvsr56_09.pdf; other longevity data are from the Human Mortality Database (HMD).

Source: Maddison (1995)

The countries noted also increased human capital investment during this period. As shown in Figure 2, the average years of education per worker in the United States, the United Kingdom, and Japan are quite close to each other—about 2 years—in 1820. While Japan stayed at this level for about half a century, the United States and United Kingdom gradually increased it to 4 years in 1870 and 6 years at the end of the 19th century. Although education levels in these two countries remained almost stationary during the 1930s and 1950s, they continued to increase soon after; the average years of education among workers in the United States and United Kingdom in 1992 were 18 and 14, respectively. Meanwhile, Japan increased its human capital investment after 1870 and has almost kept pace with the U.S. growth rate since the 1890s. In 1992 Japan achieved a level of more than 14 years of education per worker.

Figure 2: Average Years of Education per Worker in Select Countries (1820–1992)



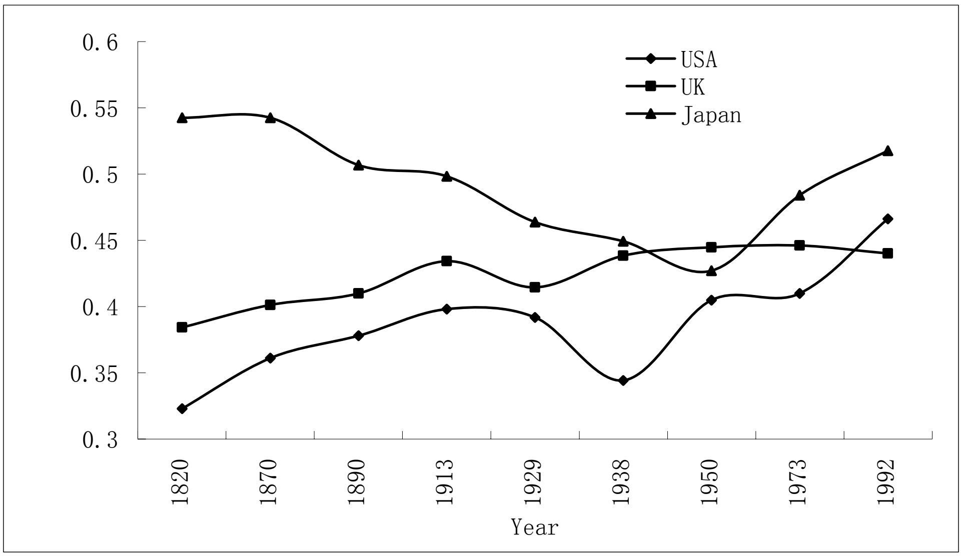
Source: Maddison (1995), Table K-1, pp. 253.

Contrary to the correlation between increased longevity and human capital investment, total labor input per capita seems to decline over time. The annual hours worked per person[1] in Japan, the United Kingdom, and the United States were, respectively, about 1,600, 1,200, and 1,070 in 1870. This is a slight increase from 1820. These working hours declined over time, to become about 970, 656, and 741 hours in 1992, indicating a 39.3%, 45.2%, and 30.8% reduction of total labor input (annual hours) in these three countries from 1870 to 1992. These trends are not special cases. Similarly declining numbers of annual hours worked per capita—with longevity increases confirmed—can be observed across European countries such as Belgium, Denmark, Finland, France, Italy, the Netherlands, Norway, Sweden, Switzerland, and in North American countries such as Canada.[2] As annual hours worked per capita can be influenced by annual hours worked per worker and by participation rates (in other words, by the employed proportion of the total population), we show the trends of annual hours worker per worker and participation rates for these three select countries in two parallel graphs (Figure 3a and Figure 3b). We find that the trend of declining total labor input per capita (as measured by annual hours worked per capita) is mainly caused by declining annual hours worked per job. As we will explain later, participation rates are determined by age structures**,,** which are mainly affected by birth rates, lifetime survival patterns, and working age intervals, while annual hours worked per job mainly relate to requirements on working hours.

Figure 3a: Participation Rates in Select Countries (1820–1992)

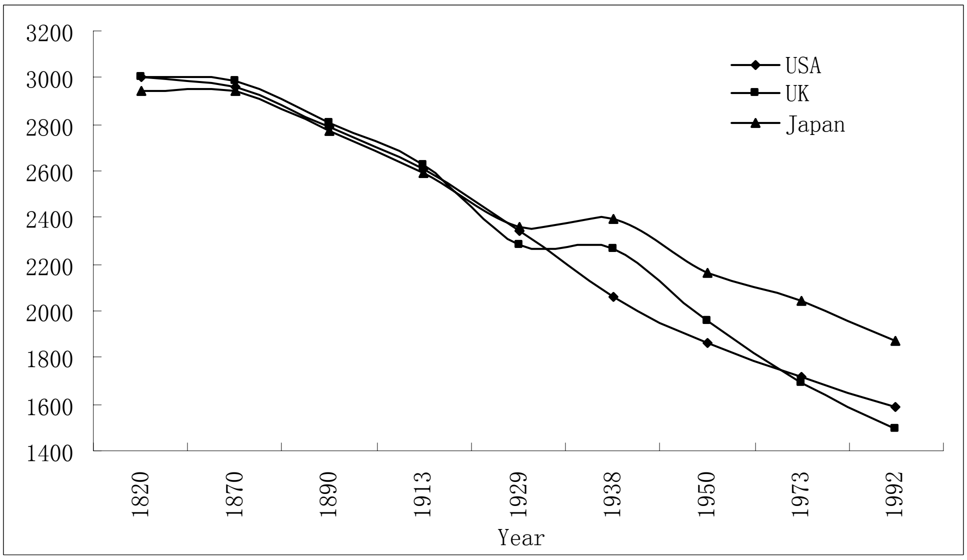
[1] Data are from Maddison (1995), Table K-1, pp. 253.

[2] According to definitions, $totallaborinputpercapita = \frac{totallaborinput(hours)}{totalpopulation} = \frac{totallaborinput(hours)}{totalemployment}\frac{totalemployment}{totalpopultion}$, we use annual hours worked per person employed (from Maddison [1995], Table J-4), total employment (from Maddison [1995], Table J-2), and total population (from Maddison (1995), Table A-3a) to calculate annual hours worked per capita. Data for life expectancies are mainly from the HMD. We also use other sources such as Johansson and Mosk (1987) and Preston, Keyfitz, and Schoen (1972) to match or complete data missing in the HMD. If data on life expectancy cannot be found in the year considered, we use data for the closest year.



Source: Calculated using data from Maddison (1995), Table K-1, pp. 253.

Figure 3b: Annual Hours Worked per Worker in Select Countries (1820–1992)



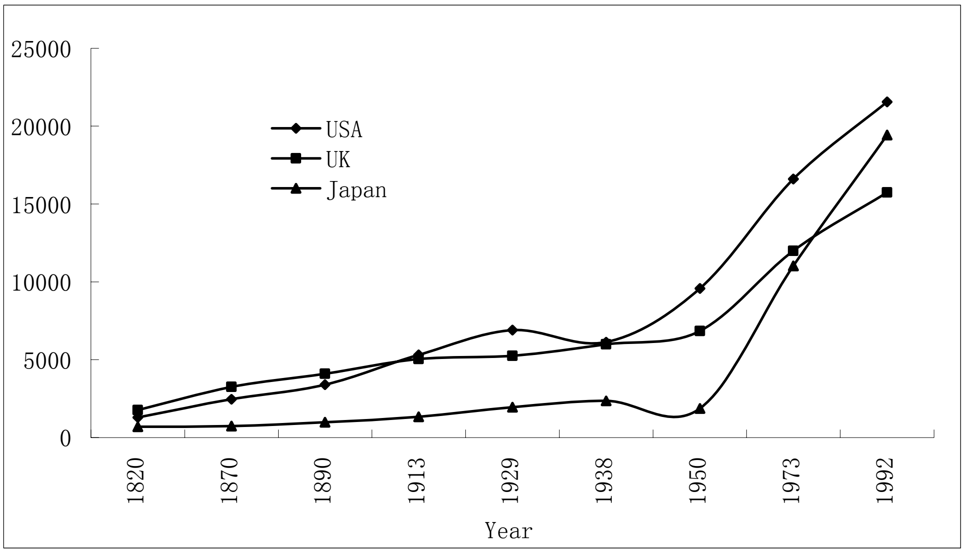
Source: Calculated using data from Maddison (1995), Table K-1, pp. 253.

The decrease in labor input between 1870 and 1992 only partially offsets the effects of increasing longevity on countries' economic growth; this may be due to a higher human capital level (as shown in Figure 2), to a higher physical capital level (especially investment in machinery and equipment),[1] and to progress in technology. In Figure 4, we show Japan

[1] Though we give no graphs here of physical investment for saving space, we give some figures to show the trends: the stocks of machinery and equipment per person employed in the United States, United Kingdom, and Japan show increases

falling behind and then catching up to the United Kingdom and United States: GDP per capita (in 1990 Geary-Khamis dollars) for these three countries in 1820 are 1,287, 1,756, and 704. Japan continues to grow much slower than the other two and falls further behind until the situation reverses after 1950, when Japan quickly catches up and even exceeds many industrialized countries (its progress praised as part of the East Asian Miracle). Figures 1, 2, and 3 may shed some light on the reasons behind this shift: Japan achieves higher gains in longevity (and the same or higher in education) and many more annual working hours, together with much higher accumulating rates of physical capital stock (as commonly cited in the literature).[1]

Figure 4: GDP per Capita in Select Countries (1820–1992)



Sources: Maddison (1995), Table D-1a, pp. 196–7.

3 Distribution of Age at Death

There is a long history of fitting models to survival and mortality experiences. Two of the

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of 140, 96, and 206 times, and the stocks of nonresidential structures per job indicate increases of 19, 13, and 61 times from 1820 to 1992 (data on Japan are from 1890 to 1992, as data from 1820 are not available; we calculate these figures using data from Table 2-2 of Maddison [1995]).

[1] See Mason (2001) for more discussions of “population change and development in East Asia.”

most popular functions are those of Gompertz (1825), which specifies an exponential rise in mortality rates with age, and Weibull, which describes the mortality rate as a power function of age. However, economic analyses of the effects of demographic changes, beginning with those of Yaari (1965) and Blanchard (1985), have typically used either an age-specific mortality rate (the infant mortality rate, IMR, may be the most common) or aggregate measurements such as life expectancy (LE); they sacrifice the details of age-dependent mortality to make analytical progress.[1] This simplification, however, may obscure the effects of demographic changes on economic development (Li and Tuljapurker, 2004). In other words, considering only a rise in life expectancy (or a change in the mortality rates of specific age groups) is not enough to reveal the relationship between improved health and economic development (Sheshinsky, 2009).

We assume the uncertainty of the age of death, T, which follows a Weibull distribution, whose range is from 0 to maximum lifetime $T_{\max}$ (allowing $T_{\max} \equiv \infty$ for simplicity here). We use a Weibull rather than a normal distribution,[2] as the former preserves the property of increasing hazard rates (we will talk a little bit more about the definition of this later) over ages, which is consistent with realistic mortality patterns and—as we will show later—both probability density $\phi(x)$ and cumulative distribution $\Phi(x)$ functions of Weibull assign zeros for all negative x, though the latter has many advantages in statistical analysis.[3]

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[1] For example, Futgami and Nakajima (2001) assume all persons die at some fixed age and study the economic effects of changes in this age; Kalemli-Ozan, Ryder, and Weil (2000) follow Yaari (1965) and Blanchard (1985) in using an exponential distribution of the survival function by assuming a fixed age-independent mortality rate.

[2] As one of the earliest studies to use realistic mortality patterns for the United States, Li and Tuljapurker (2004) use a normal distribution for their analytical analysis of the effects of demographic changes on economic growth.

[3] Zhang (2010), using the nonlinear estimating method and data from China's 2000 population census at the county level (2,870 observations), finds that both the Weibull (with two parameters, where shape and scale are separately estimated) and the normal distribution can fit actual mortality patterns quite well (though Weibull seems more favorable as it captures the somehow long left tail and sharp decline trend of dying probabilities at ages older than the peak age at death). The shape parameter is estimated to be larger than 6 in most cases (this is also confirmed with data from other countries such as the United States, the United Kingdom, Sweden, and Japan); therefore, Weibull and normal distributions are statistically

A general Weibull distribution has three parameters: location, scale, and shape. As the main properties examined in this article are independent of the location parameter, we consider the distribution with the location parameter is equal to zero, which reduces it to the two-parameter form commonly used. Therefore the probability density function (PDF) and the cumulative distribution function (CDF) of the Weibull distribution over age groups are, respectively:

$$\phi\left(a;\lambda,k\right)=\frac{\lambda}{k^{\lambda}}a^{\lambda-1}e^{-\left(\frac{a}{k}\right)^{\lambda}} \qquad (1)$$

$$\Phi\left(a;\lambda,k\right)=1-e^{-\left(\frac{a}{k}\right)^{\lambda}} \qquad (2)$$

where λ is the shape parameter, k is the scale parameter, and λ and k are strictly positive.

When the age at death follows a Weibull distribution, the associated life expectancy and the variance of age at death are

$$LE=k\Gamma\left(1+\frac{1}{\lambda}\right);\ \ \sigma^{2}=k^{2}\left[\Gamma\left(1+\frac{2}{\lambda}\right)-\left(\Gamma\left(1+\frac{1}{\lambda}\right)\right)^{2}\right]$$

where $\Gamma\left(z\right)=\int_{0}^{\infty}x^{z}e^{-x}dx$ is a gamma function.

The associated hazard rate (or age-specific mortality rate) is defined as a conditional probability of dying at age a ,

$$HR\left(a;\lambda,k\right)=\frac{\phi\left(a\right)}{1-\Phi\left(a\right)}=\frac{\lambda}{k^{\lambda}}a^{\lambda-1} \qquad (3)$$

From equation (3), we derive that hazard rates increase with age if $\lambda>1$ or are constant if λ equals 1, the latter indicating an exponential distribution, which is also often used.[1]

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different in this situation (applied statisticians suggest when the shape parameter of a Weibull distribution is approximately chosen to be around 3.5, then Weibull and normal distributions can be statistically equivalent [Makino, 1984]).

[1] Exponential distribution is easier to use when it generates constant age-specific mortality rates over ages. Boucekkine, Croix, and Licandro (2002) estimate that a density function of age at death always increases with age. This property of density function is also assumed by Sheshinsky (2009), both of which contradict the empirical situation in which a hump

Gains in longevity can be achieved through changes in k or λ or both. While Li and Tuljapurker (2004), among others, find historical evidence for growing life expectancy—indicating the rightward shift of a more leptokurtic distribution of the death age— Zhang (2010), among others, demonstrates that the estimated shape parameter $\hat{\lambda}$ is much bigger than 1. In this case, increasing λ (keeping k unchanged) when $\lambda > 1$ alone can depict these patterns of the distribution of the death age over time. For the sake of simplicity, we will focus on changes to λ.

The Weibull approximation may fail to capture two aspects of historical death rates: high death rates at very early ages (between 0 and 1), and the linear negative relationship between life expectancy and the variance of death age.[1]

4 Life-cycle Model with Exogenous Overtime Longevity Risks

Suppose that at any given time *t*, the random death age *T* has a known distribution; the PDF is $\phi_t(a)\,(\equiv \phi(a;\lambda_t,k))$ and related survival function $l_t(a)\,(l_t(a)=\int_a^\infty \phi_t(x)\,dx$. For any given time *t*, we first define an aggregate function as

$$g_t(z) \equiv E_T\left[e^{zT}\right] = \int_0^\infty e^{za}\phi_t(a)da = \sum_{i=0}^{\infty}\frac{z^i\lambda^i}{i!}\Gamma\left(1+\frac{i}{k}\right) \quad (4)$$

which is known as the moment-generating function of *T* whenever this expectation exists. And we can calculate the period life expectancy LE_t and the variance of age at death σ_t^2 at time *t* by

$$LE_t = \frac{dg_t(z)}{dz}|_{z=0}; \quad \sigma_t^2 = \frac{d^2 g_t(0)}{dz^2}|_{z=0} - LE_t^2$$

shape with a well-defined peak is found (see Wilmoth and Horiuchi, 1999).

[1] This negative relationship is well defined in Wilmoth and Horiuchi (1999) and Li and Tuljiapurker (2004). Zhang (2010) uses data from China's 2000 population census at the county level (2,870 observations) and also confirms this negative relationship.

Thereafter, we define individuals at time t with the same period distribution ϕ_t as generation *t;* we thus use t to comprehensively represent this generation.[1]

Representative t assumes that a person will spend ages 0 to E_t in education; but as the person may even die before finishing schooling, we define the number of expected years in school to be (see Appendix[i])

$$\chi_t(E_t) \equiv \int_0^{a_s} l_t(a)da = E_T\left[T \wedge E_t\right] \quad (5)$$

Obviously $\chi_t(E_t)$ is often a nondecreasing function of E_t, $\chi_t(\infty) \equiv E_T\left[T\right] = LE_t$, and $\chi_t(0) = 0$.

Assume that representative t provides 1 unit of labor at each age when working (individuals work until retiring at age R_t[2] after they finish schooling and for as long as they are alive). We can similarly derive representative t's expected working years $W_t\left(E_t, R_t\right)$ and, at any time t, generation t's participation rate $m_t\left(E_t, R_t\right)$:

$$W_t\left(E_t, R_t\right) \equiv \int_{E_t}^{R_t} l_t\left(a\right)da = \chi_t\left(R_t\right) - \chi_t\left(E_t\right) \quad (6)$$

$$m_t\left(E_t, R_t\right) \equiv \frac{W_t\left(E_t, R_t\right)}{LE_t} = \frac{\chi_t\left(R_t\right) - \chi_t\left(E_t\right)}{LE_t} \quad (7)$$

with an increase in R_t and decrease in E_t as intuited. And if there are no retiring requirements, then

$$m_t(E_t) \equiv \lim_{R_t \to \infty} m\left(E_t, R_t\right) = 1 - \frac{\chi_t\left(E_t\right)}{LE_t}.$$

We denote representative t's consumption at age a by $c_t(a)$, given exogenous demographic patterns (a series of distributions of age at death). With no bequest motive, the

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[1] Here we get this definition from that of period life expectancy. To clarify, the period life expectancy at birth (PLE_0) for a given year represents the average number of years of life remaining if a group of persons were to experience *the mortality rates for that year* over the course of their remaining lives. Meanwhile, the cohort life expectancy at birth (CLE_0) for a given year represents the average number of years of life remaining if a group of persons were to experience *the mortality rates for the series of years* in which they reach each succeeding age.

[2] We assume that the retirement age is exogenously set and known to every individual at time *t*. Here we abstract from the optimal retirement age and simply assume an exogenous retirement age. Refer to Sheshinsky (2009) for more detailed discussions of retiring decisions for which a disutility function of work is introduced.

expected utility of the infinite social planner is

$$\int_0^\infty N_t e^{-\beta t} \int_0^\infty l_t(a) u(c_t(a)) e^{-\theta a} da dt$$

where β is the time-discounting factor, as commonly used (which is often set to be 0.96 in discrete cases; here we assume it to be 0.04 as for the continuous situation). The above utility function also implies decreasing (θ is positive) or increasing (θ is negative) or equal (θ is 0) weights that the social planner pays to individuals over various ages.

We assume that the aggregate output follows the Cobb-Douglas production function

$$Y_t = AK_t^\alpha (A_t H_t)^{1-\alpha}$$

where K_t and H_t are aggregate physical and human capital at time *t*, respectively. A_t is total exogenous labor productivity,[1] which is assumed to grow at a rate of g. We will define and describe H_t in detail later. Meanwhile, the accumulation of physical capital K_t satisfies

$$\dot{K}_t = Y_t - N_t \int_0^\infty c_t(a) l_t(a) da - \delta K_t .$$

Hereafter we specify a constant relative risk aversion (CRRA) utility function for each individual:

$$u(c_t(a)) = \frac{c_t(a)^{1-\gamma}}{1-\gamma}$$

where γ is the (time-invariant) coefficient of relative risk aversion and is the same for all individuals.

The social planner chooses inter-temporal allocations of physical capital and consumption among generations, levels of educational attainment for each generation, and inner-temporal allocations of consumption among individuals with various ages within the

[1] With the definitions and forms here, one may assume that A_t includes all other factors affecting labor productivity other than educational attainment.

same generation, to maximize the expected utility. The utility maximizing problem (UMP) that follows is therefore[1]

$$
\begin{aligned}
&\max \int_0^\infty N_t e^{-\beta t} \int_0^\infty l_t(a) \frac{c_t(a)^{1-\gamma}}{1-\gamma} e^{-\theta a} da dt \\
&s.t. \quad Y_t = AK_t^\alpha (A_t H_t)^{1-\alpha} \qquad (8) \\
&\dot{K}_t = Y_t - N_t \int_0^\infty c_t(a) l_t(a) da - \delta K_t
\end{aligned}
$$

We solve (8) in two steps. In Step 1, which means at time *t*, given the inter-temporal arrangement of physical capital $\{K_t\}_{t\geq 0}$ and consumption $\{c_t\}_{t\geq 0}$, where c_t is generation *t*'s consumption per capita at time *t*, the social planner chooses optimal inner-temporal allocation $\{c_t(a)\}_{a\geq 0}$ to maximize representative *t*'s utility:

$$
\begin{aligned}
&\max_{\{c_t(a)\}_{a\geq 0}} \int_0^\infty l_t(a) u(c_t(a)) e^{-\theta a} da \\
&s.t. \quad c_t = \int_0^\infty l_t(a) c_t(a) da \qquad (BC_t)
\end{aligned} \qquad (9)
$$

The standard optimality conditions of (9) yield optimal inner-temporal allocations $\{c_t(a)\}_{a\geq 0}$ (see appendix[ii]):

$$c_t(a) = e^{-\frac{\theta}{\gamma}a} c_t(0)$$

and together with the budget constraint (BC_t) yield representative *t*'s consumption at birth $c_t(0)$:

$$c_t(0) = \frac{\theta}{\gamma} \frac{c_t}{1 - g_t(-\frac{\theta}{\gamma})}$$

Thus $c_t(a)$ is a decreasing function of a if θ is positive, as intuited by the assumption of the social planner's decreasing weights over ages.

And at any given time *t*, the maximizing utility for the representative *t* is

[1] We introduce no cost for education, similar to Sheshinsky (2009).

$$\int_0^\infty l_t(a)u(c_t(a))e^{-\theta a}da=\int_0^\infty l_t(a)\frac{\left[\frac{\frac{\theta}{\gamma}e^{-\frac{\theta}{\gamma}a}}{1-g_t(-\frac{\theta}{\gamma})}c_t\right]^{1-\gamma}}{1-\gamma}e^{-\theta a}da=\left[\frac{1-g_t(-\frac{\theta}{\gamma})}{\frac{\theta}{\gamma}}\right]^{\gamma}\frac{c_t^{1-\gamma}}{1-\gamma}.$$

In step 2, inter-temporal allocation, since the utility function can preserve its properties after monotonic transformation, we compute

$$\max\int_0^\infty N_t e^{-\beta t}\left[\frac{1-g_t(-\frac{\theta}{\gamma})}{\frac{\theta}{\gamma}}\right]^{\gamma}c_t^{1-\gamma}dt\Leftrightarrow\max\int_0^\infty N_t e^{-\beta t}\left[1-g_t(-\frac{\theta}{\gamma})\right]^{\gamma}\frac{c_t^{1-\gamma}}{1-\gamma}dt.$$

Thus (8) is equivalent to

$$\begin{aligned}
&\max_{\{K_t\},\{c_t\},\{a_s(t)\}}\int_0^\infty N_t e^{-\beta t}\left[1-g_t(-\frac{\theta}{\gamma})\right]^{\gamma}\frac{c_t^{1-\gamma}}{1-\gamma}dt\\
&s.t.\quad Y_t=AK_t^{\alpha}(A_tH_t)^{1-\alpha}\\
&H_t=\left[\frac{\chi_t(R_t)-\chi_t(E_t)}{LE_t}\right]N_t f(E_t)\\
&\dot{K}_t=Y_t-N_tc_t-\delta K_t
\end{aligned}\tag{10}$$

Here, as we assume the relative working productivity of individuals depends only on their years of schooling (i.e. $f(E_t)$, where $f(0)$ is normalized to 1), with this assumption the aggregate human capital at time t is just $f(E_t)L_t$, where $L_t=m_t(E_t,R_t)N_t=\left[\frac{\chi_t(R_t)-\chi_t(E_t)}{LE_t}\right]N_t$ is the total labor force at time *t*, as generation *t*'s effective labor input per capita is $m_t(E_t,R_t)$.

Let $k_t=\frac{K_t}{A_tN_t}$, $h_t=\frac{H_t}{N_t}$, $y_t=\frac{Y_t}{A_tN_t}$, $\xi(t)=\frac{c_t}{A_t}$ (effective physical capital per capita, human capital per capita, output per capita, and consumption per capita). Together with assumed population and technology growth rates, $\frac{\dot{N}_t}{N_t} \equiv n$ and $\frac{\dot{A}_t}{A_t} \equiv g$, (7) can be solved by

$$\begin{aligned}
&\int_0^\infty e^{-\hat{\beta}t}\left[1-g_t(-\frac{\theta}{\gamma})\right]^\gamma \frac{\xi_t^{1-\gamma}}{1-\gamma}dt \\
&s.t. \quad y_t = Ak_t^\alpha h_t^{1-\alpha} \\
&h_t = \left[\frac{\chi_t(R_t)-\chi_t(E_t)}{LE_t}\right] f(E_t) \\
&\dot{k}_t = y_t - \xi_t - (n+g+\delta)k_t
\end{aligned} \qquad (11)$$

where $\hat{\beta} = \beta - n - (1-\gamma)g$.

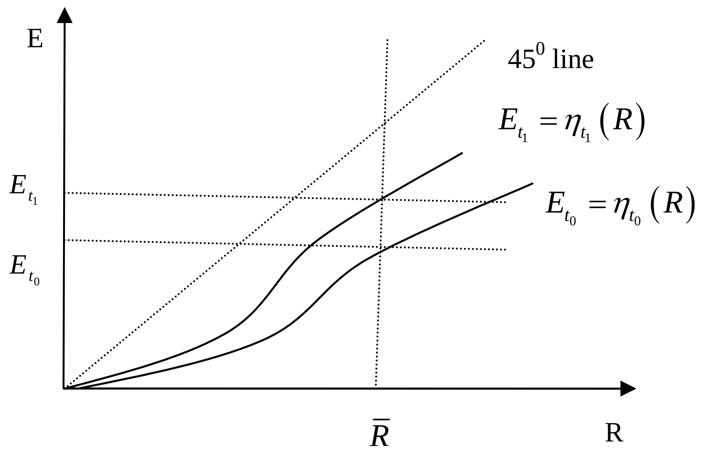
At any given time *t*, the optimal education E_t^* satisfies (see appendix[iii]):

$$\int_{E_t}^{R_t} f'(E_t)\frac{l_t(a)}{l_t(E_t)}da - f(E_t) = 0 \quad (12)$$

In the optimal scenario, the marginal benefit of an additional lifetime investment in education for a representative of generation *t*, which equals the conditional expected increase in a person's lifetime relative wage rate (LHS), is equal to the current marginal income loss (RHS) due to one less unit of time in labor input.[1] Equation (12) also means that at any time *t*, the optimal education E_t^* is a function of R_t. We denote this relationship to be $E_t^* = \eta_t(R_t)$. Generally this relationship depends on the distribution of age at death, and, as will be concluded in Proposition 1, the function η_t is an upward sloping curve in the (E,R) planes with $\eta_t(0) \equiv 0$ and with an upper bound of 45^0 line as E_t must strictly less than R_t (we show a possible shape in Figure 5, even though we specify no particular form of $f(\bullet)$ here).

Figure 5: Optimal Education Length and Retirement Age

[1] Since we assume that "the relative working productivity (hence relative wage rate) of individuals only depends on their years of schooling" (Just as Li & Tuljapurker (2004)) for any generation, representative *t*'s relative wage rate $\omega_t(a)$ satisfies the following step function: $\omega_t(a) \equiv \begin{cases} f(E_t)w_t & a \in [E_t, R_t] \\ 0 & elsewhere \end{cases}$, where w_t is the baseline wage rate, which is equal to the marginal change in aggregate output with respect to aggregate human capital for the generation *t*. Therefore for any subset age interval of $[E_t, R_t]$, the wage earned by representative *t*, if alive, equals his/her relative wage rate multiplied by the length of the interval, which makes this conclusion hold, though we omit w_t on both sides.



Proposition 1[1]: If $f(\bullet)$ is strictly increasing and strictly concave, which assumes diminishing positive returns to schooling, an increase in longevity will induce an increase in schooling. (Proof: see appendix[iv].)

With an optimal condition (12) for exterminating optimal education and definitions of (7), we then derive the participation rate at any time t as $m_t(E_t, R_t) = \frac{l_t(E_t)\frac{f(E_t)}{f'(E_t)}}{LE_t}$. The following proposition provides conditions of changes of participation rate in accord with increasing life expectancy.

[1] More realistic scenarios that include increased life expectancy can be discussed here with (9) if we relax the assumption of a Weibull distribution and understand that this proposition may not hold. Suppose these three different scenarios of survival probabilities: (1) LE increases mainly due to declining mortality rates of very young or old age groups, which keeps $\frac{l_t(a)}{l_t(E_t)}$ (where $a \in [E_t, R_t]$) nearly unchanged (in the first case both $l_t(a)$ and $l_t(E_t)$ increase multiplicatively, whereas they remain unchanged in the latter), with conditions indicated by (9), we will find E_t nearly unchanged; (2) suppose LE increases mainly due to the declining mortality rates of middle age groups, which makes $\int_{E_t}^{R_t} l_t(a)\,da$ increase much more than $l_t(E_t)$, E_t will increase to make (9) hold again. As cited in Sheshinsky (2009) there are "three phases of gains in health from declining mortality rates of different age groups in modern history: in the early 20th century, infant survival probabilities improved dramatically, followed in mid century by major improvements of survival probabilities in middle-ages (50–70) due to the cardiovascular revolution. During recent decades, improvements of life prospects due to new medical technologies focused mainly on the old and the very old." We then know that higher longevity may lead to different directions of educational attainment. Maybe that is why we see a remarkable increase in years of education for select countries in Figure 2, or we would predict somehow faster increases in educational attainment for those countries where gains of higher life expectancy are mainly from declining mortality rates of middle-age groups.

Proposition 2: Even if $f(\bullet)$ is strictly increasing and strictly concave, an increase in longevity that induces an increase in schooling cannot warrant an increase in labor input.

Proof: Though $\frac{f(E_t)}{f^{'}(E_t)}$ increases in E_t, $l_t(E_t)$ is indeterminate in E_t since both the schooling age and survival curve shift toward the right; therefore, the $m_t(E_t, R_t)$ increase or decrease of higher life expectancy depends on how prolonging longevity is achieved through changes in lifetime survival patterns.

Similar to the definition of h_t, we then get $h_t = m_t(E_t, R_t) f(E_t) = \frac{l_t(E_t)\left[f(E_t)\right]^2}{f^{'}(E_t) LE_t}$, though we still cannot determine whether h_t increases or decreases in response to changes in LE_t. Hence we find that after considering the effects of progress on other factors of labor productivity, effective human capital per capita (depending on education) is indeterminate when life expectancy grows.

The standard optimality conditions yield optimal inter-temporal allocations $\{\xi_t\}_{t\ge 0}$, $\{k_t\}_{t\ge 0}$

$$\frac{\dot{\xi}_t}{\xi_t} = \frac{r_t - (n+g+\delta) - \hat{\beta}}{\gamma} - \frac{\dot{g_t}(-\frac{\theta}{\gamma})}{1 - g_t(-\frac{\theta}{\gamma})} \quad (13)$$

where $r_t = \frac{\alpha y_t}{k_t} = \alpha A k_t^{\alpha-1} h_t^{1-\alpha}$.

The first-order Taylor expansion when $\frac{\theta}{\gamma}$ is small[1] is,

$$g_t(-\frac{\theta}{\gamma}) \approx 1 - \frac{\theta}{\gamma} LE_t$$

[1] Since $\theta = 0.03$ is commonly used (for example, by Kalemli-Ozcan, Ryder, and Weil [2000]) for calibration, and $\gamma \in [2,10]$ is suggested in finance (Zhou, 2007), $\gamma = 1$ as a macroeconomic model often assumes a log utility function to ensure the existence of a balanced growth path.

Hence, with $\frac{\dot{g_t(-\frac{\theta}{\gamma})}}{1-g_t(-\frac{\theta}{\gamma})} = -\frac{\dot{LE_t}}{LE_t}$ we then get the following Euler Equation (EE):

$$\frac{\dot{\xi_t}}{\xi_t} = \frac{r_t - (n+g+\delta) - \hat{\beta}}{\gamma} + \frac{\dot{LE_t}}{LE_t} \quad (14)$$

Equation (14) is under the uncertainty of survivorship over ages. Comparing this EE to the classical EE,

$$\frac{\dot{\xi(t)}}{\xi(t)} = \frac{R_t - (n+g+\delta) - \hat{\beta}}{\gamma} \quad (15)$$

where $R_t = \alpha A k_t^{\alpha-1}$, and $\xi(t)$ is, similar as ξ_t, effective consumption per capita. We find that increasing longevity will affect the growth rate of consumption through two mechanisms. The first, the "direct effect" (the second term of RHS in (14)), is positive, as the social planner is willing to care for future generations more because of their longer lifetimes. This direct effect can be totally separate from the coefficient of relative risk aversion γ and time-discounting factor β. This may be because the perfect risk-sharing system for each generation irons out any uncertainty regarding the longevity of individuals of the same generation, which is exactly why we treat individuals with the same Weibull distribution as the same generation. Second, the "indirect effect" (the first term of RHS in [14]) depicts how an increase in longevity influences the marginal return on effective physical capital per capita. By analyzing effective human capital per capita, we see if generations increase their labor input over time as life expectancy grows. This would induce a further growth of consumption, though, for now, it is more valuable to reduce current consumption in exchange for future higher output. This is summarized in Proposition 3.

Proposition 3: An increase in longevity alone may not warrant higher rates of investment in human and physical capital, depending on whether the direct positive effect or indirect (possible) negative effect dominates.

From Proposition 3 we find that a steady state cannot be necessarily derived as investment in human capital; therefore, investment in physical capital is indeterminate. We here only consider a very special situation in which $\frac{\dot{LE}_t}{LE_t} \equiv \upsilon$ and $h_t \equiv h$ holds for all time t (in this case we also can simply assume h to be 1). We then derive the steady level of effective output per capita to be

$$y_t = Ak_t^{\alpha} = A\left[\frac{\alpha A}{\delta+\beta+\gamma(g-\upsilon)}\right]^{\frac{\alpha}{1-\alpha}} \equiv y^* .$$

And at the balanced growth path, the capital (K_t) and output (Y_t) per capita grow at the same rate of $n+g$, the generation t's consumption per capita (c_t) grows at g. Hence we find that prolonging longevity can increase the levels of effective capital ($k^* = \left[\frac{\alpha A}{\delta+\beta+\gamma(g-\upsilon)}\right]^{\frac{1}{1-\alpha}}$) and output ($y^*$) per capita, as both k^* and y^* increase in υ, while have no effect on the balanced growth rate of the economy.

Further, a change in υ leads to the same direction and a more sensitive change in y^*, as

$$\frac{dy^*}{d\upsilon} = A(\alpha A)^{\frac{\alpha}{1-\alpha}} \left(\delta+\beta+\gamma(g-\upsilon)\right)^{\frac{1}{\alpha-1}} \frac{\gamma\alpha}{1-\alpha} > 0$$

$$\frac{d^2 y^*}{d\upsilon^2} = A(\alpha A)^{\frac{\alpha}{1-\alpha}} \frac{\gamma^2\alpha}{(1-\alpha)^2} \left(\delta+\beta+\gamma(g-\upsilon)\right)^{\frac{2-\alpha}{\alpha-1}} > 0$$

which means that some countries will take off if they take the lead in increasing their life expectancy at some point in time, increasing their edge over countries that are still experiencing an initial lower growth rate in longevity. This may explain why economic miracles occur in these countries and not in their counterparts. As countries with initial lower life expectancies usually have higher potential growth rates of longevity, this may explain why countries will converge in many cases.

5 Conclusions

Many researchers have concluded that longer life expectancies promote investment in education, as a prolonged labor supply raises the rate of return on education. Besides explaining the empirical evidence behind this conclusion (at an absolute level), there is another issue to be discussed: whether relative time spent in studying and working would increase in growing life expectancies. Existing conclusions in literature are derived from overlapping-generation (OLG) models, and rely on strict while unrealistic assumptions on life cycle mortality rates: these OLG models only consider changes of mortality rates of some specific ages (usually mortality rates of the old as in two-period or three-period OLG models), which means they only consider the neglect the effects of changes in mortality of other ages (such as immature deaths). The problems of these models are that they would be more preferable and more consistent with the aging economies which just emerge, and they need to link changes in longevity one-to-one with changes in those mortality rates which is wildly unrealistic rather than the overall life cycle survival patterns, and they could only consider the effects of absolute changes in longevity on economic development.

We combine methods and insights from Li and Tuljapurker (2004) and Sheshinsky (2009) in this article. That said, it diverges from Li and Tuljapurker (2004) by using a Weibull distribution, which is more consistent with realistic mortality patterns than a normal distribution. It also differs from these two studies as it considers interactions across generations. We are motivated by conclusions from Aghion, Howitt, and Murtin (2010) and Zhang (2010) that both higher initial life expectancies and steeper increases in longevity have significant positive effects on income growth, and try to fill the gap to directly theoretically link growth rate of life expectancy with economic growth.

We then find that we can still keep the relationships among longevity and human capital investment and labor input, and explain gaps among existing hypotheses and between these hypotheses and empirical evidence, by using a more realistic extended life-cycle model. We demonstrate that prolonged longevity prompts increased investment in education but does not increase labor input. Instead, labor input depends on how life expectancies increase due to changes in lifetime survival probabilities over generations. We also demonstrate that an increase in life expectancy may lead to different decisions on investment in education and lifelong labor input; therefore, dynamic consumption, accumulation of physical investment, and economic development depend on changes in the hazard rate at different ages for the same generation and changes in hazard rates at the same age over several generations. We show that prolonged life expectancy will cause individuals to increase their human capital investment, but that it may not warrant rises in labor input. Later we find that higher improvement rate of longevity will promote economic growth, even we exclude the mechanism of human capital formation, and only consider growth effects of higher improvement rate of life expectancy from physical capital investment.

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i According to $l_t(a) = \int_a^\infty \phi_t(x)\,dx$,

$$\begin{aligned}
E_T[T \wedge a_s] &= \int_0^{a_s} a\phi_t(a)da + \int_{a_s}^\infty a_s\phi_t(a)da \\
&= \int_0^{a_s} ad(-l_t(a)) + a_s\int_{a_s}^\infty d(-l_t(a)) \\
&= -al_t(a)\Big|_0^{a_s} + \int_0^{a_s} l_t(a)da - a_s\, l_t(a)\Big|_{a_s}^\infty \\
&= \int_0^{a_s} l_t(a)da
\end{aligned}$$

ii The according Hamiltonian function is

$$H = l_t(a)\frac{(c_t(a))^{1-\gamma}}{1-\gamma}e^{-\theta a} + \lambda_a\left(c_t - \int_0^\infty l_t(a)c_t(a)da\right)$$

First-order condition (FOC)

$$\begin{aligned}
&H_c = 0 \\
&\Rightarrow l_t(a)(c_t(a))^{-\gamma}e^{-\theta a} - \lambda_a l_t(a) = 0 \\
&\Rightarrow c_t(a) = e^{-\frac{\theta}{\gamma}a}\lambda_a^{-\frac{1}{\gamma}}
\end{aligned}$$

By replacing $c_t(a) = e^{-\frac{\theta}{\gamma}a}\lambda_a^{-\frac{1}{\gamma}}$ into (BC_t), we then get $\lambda_a^{-\frac{1}{\gamma}} = \frac{\theta}{\gamma}\frac{c(t)}{1-g_t(-\frac{\theta}{\gamma})}$,

and $c_t(a) = \dfrac{\frac{\theta}{\gamma}e^{-\frac{\theta}{\gamma}a}}{1-g_t(-\frac{\theta}{\gamma})}c(t)$.

iii The subsequent Hamiltonian function is

$$H = e^{-\hat{\beta}t}\left[1-g_t(-\frac{\theta}{\gamma})\right]^\gamma \frac{\xi_t^{1-\gamma}}{1-\gamma} + \lambda_t\left\{Ak_t^\alpha h_t^{1-\alpha} - \xi_t - (n+g+\delta)k_t\right\}$$

FOCs are:

FOC1:

$H_{E_t}=0 \Rightarrow \chi_t^{'}(E_t)f(E_t)=\left[\chi_t(R_t)-\chi_t(E_t)\right]f^{'}(E_t)$, and together with a definition of $\chi_t(a)$ ($\chi_t(a)=\int_0^a l_t(a)da$), we then get $\int_{E_t}^{R_t} f^{'}(E_t)\frac{l_t(a)}{l_t(E_t)}da = f(E_t)$.

FOC2:

$$H_c=0$$

$$\Rightarrow \lambda_t = e^{-\hat{\beta}t}\left[1-g_t(-\frac{\theta}{\gamma})\right]^{\gamma}\xi(t)^{-\gamma}$$

FOC3:

$$H_k=-\dot{\lambda}_t$$

$$\Rightarrow \lambda_t\left(r_t-n-g-\delta\right)=-\left\{e^{-\hat{\beta}t}\left[1-\dot{g}_t(-\frac{\theta}{\gamma})\right]^{\gamma}\xi_t^{-\gamma}\right\}=\lambda_t\left\{\hat{\beta}+\gamma\frac{\dot{g}_t(-\frac{\theta}{\gamma})}{1-g_t(-\frac{\theta}{\gamma})}+\gamma\frac{\dot{\xi}_t}{\xi_t}\right\}$$

$$\Rightarrow \frac{\dot{\xi}_t}{\xi_t}=\frac{r_t-n-g-\delta-\hat{\beta}}{\gamma}-\frac{\dot{g}_t(-\frac{\theta}{\gamma})}{1-g_t(-\frac{\theta}{\gamma})}\text{, where } r_t=\frac{\alpha y_t}{k_t}=\alpha A k_t^{\alpha-1}h_t^{1-\alpha}.$$

[iv] First, by totally differentiating (9), we can derive

$$\eta_t^{'}(R_t)=\frac{\partial E_t}{\partial R_t}=\frac{l_t(R_t)f^{'}(E_t)}{l_t^{'}(E_t)f(E_t)+2l_t(E_t)f^{'}(E_t)-\left[\chi_t(R_t)-\chi_t(E_t)\right]f^{''}(E_t)}>0,$$

as $f^{'}(E_t)>0$ and $f^{''}(E_t)<0$, which explains the positive relationship between E_t and R_t.

Second, we denote the equation (12) for optimal education

as $\psi(E_t,R_t;\lambda_t)\equiv\int_{E_t}^{R_t} f^{'}(E_t)\frac{l_t(a)}{l_t(E_t)}da-f(E_t)=0$, where $l_t(a)\equiv l(a;\lambda_t)$, and when the distribution of age at death follows the Weibull distribution, the hazard rate for representative *t* (any *t*) strictly increases over age groups, which indicates a ψ shifts upward and to the left when life expectancy increases (in this case λ decreases), as we neglect the subscript t for simplicity here.

$$\psi_\lambda = f^{'}(E)\left\{\frac{\left[\int_E^R \frac{\partial l(a;\lambda)}{\partial \lambda}da\right] l(E;\lambda) - \left[\int_E^R l(a;\lambda)\,da\right]\frac{\partial l(E;\lambda)}{\partial \lambda}}{\left[l(E;\lambda)\right]^2}\right\}$$

$$= f^{'}(E)\left\{\left[\int_E^R \frac{\frac{\partial l(a;\lambda)}{\partial \lambda}}{l(a;\lambda)}\frac{l(a;\lambda)}{l(E;\lambda)}da\right] - \left[\int_E^R \frac{\frac{\partial l(E;\lambda)}{\partial \lambda}}{l(E;\lambda)}\frac{l(a;\lambda)}{l(E;\lambda)}da\right]\right\}$$

$$= \frac{f^{'}(E)\int_E^R l(a;\lambda)\,da}{l(E;\lambda)}\int_E^R\left[\frac{\frac{\partial l(a;\lambda)}{\partial \lambda}}{l(a;\lambda)} - \frac{\frac{\partial l(E;\lambda)}{\partial \lambda}}{l(E;\lambda)}\right]\frac{l(a;\lambda)}{\int_E^R l(a;\lambda)\,da}da$$

$$= \frac{f^{'}(E)\int_E^R l(a;\lambda)\,da}{l(E;\lambda)}\int_E^R\left[HR(E;\lambda) - HR(a;\lambda)\right]\frac{l(a;\lambda)}{\int_E^R l(a;\lambda)\,da}da$$

$$< 0$$

, as $HR(a;\lambda_t) \equiv HR(a;\lambda_t,k) = \frac{\lambda}{k^\lambda}a^{\lambda-1}$ increases over ages when $\lambda_t > 1$. See more discussions in Sheshinsky (2009). Hence when the retirement age is fixed at some level, an increase in education will increase life expectancy. In addition, if the retirement age is postponed as citizens live longer, this will further induce an increase in human capital investment.