COOPERATION AND SUCCESS IN PLURALISTIC SOCIETIES

AN ANALYTIC MATHEMATICAL APPROACH

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Abstract Questions of cooperation, peace and success within a society have been frequently modelled by iterated (2×2)-bimatrix-games and sets of strategies for such games. In most of the research up to now computer simulations have been used to understand what would happen in somehow idealized societies. This paper introduces an approach relying solely on analytic methods. It is based on the assumption, that the strategies are chosen according to some distribution function. As a first instance of a case, where an explicit solution can be obtained, the case of uniform distribution will be studied.

Consider a society, where there is no a priori preference for any particular strategy. What sort of behavior will be successful in such a society? Will there be a 'natural ruler' for such a society and will this person be trustworthy? Will the well-known candidates for good strategies, such as TIT FOR TAT, show a tendency to cooperation, when they are used against many different opponents? The following main results will be obtained: Assuming uniform distribution there is a strategy, which is in the integral-mean best among all memory-one-strategies. This strategy is – against it's myth – not TIT FOR TAT, because PAVLOV does better. The payoff of TIT FOR TAT in the integral-mean is calculated explicitly

Thomas Hobbes' 'state of nature' has been the starting point for many considerations concerning social order and legitimation of sovereignity. Philosophers, sociologists, mathematicians and economists have asked whether it is possible, that social order emerges without any external force, only as a result of the rational preferences and deliberations of individuals. We are hence confronted with two questions: What sort of behavior does a rational person choose? Are rational persons able to live in peace with each other? At first sight these questions might seem independent, but it is clear, if we assume that physical and mental abilities are distributed fair among all the individuals[1], then there is complete symmetry among these persons and therefore one's optimal strategy will be everybody's optimal strategy. Then it is of course easily predictable, if such a society will attain a state of cooperation and peace. But

[1]Hobbes is one the first who makes this premise, when he writes in *Leviathan* ([7], p.183): Nature hath made men so equal, in the faculties of body, and mind.

unfortunately there is no 'optimal' behavior in a sense, that supposes itself. Axelrod proved this for one specification of 'optimal', and this paper contains a proof for another specification. So what we are left with, is: *The* rational person/behaviour does not exist.

Therefore we have to try a different attempt: If there is no optimal strategy, it is natural to think of every individual just choosing *any* strategy. Let us assume the strategies to be distributed uniformly, so that the society could be called totally pluralistic with respect to the choice of strategies. What strategy is then best against these strategies in the mean? In other words: What properties does a natural ruler, a person qualified by her/his success, have[2]?

In the following I will discuss two strategies and ask whether they are successful and whether they are cooperative strategies.

0 Game-theoretic modelling and the existence of optimal strategies

In a world that doesn't consist of angels only, as Axelrod [1] remarked, it is of interest to know, under what conditions individuals will cooperate, when they are supposed to be set in a state of nature without any executed law.

Game theorists have modelled the decision making process in the state of nature by the so called Prisoner's Dilemma, more generally by (2×2)-bimatrix-games. These models have become common knowledge to all persons dealing with problems of (social) cooperation or social order.

For the purpose of building a game-theoretic model for the state of nature let us assume two players S_1 and S_2 who may choose (independently of each other) between two alternatives C and D. After having decided they gain their payoff according to the following matrix

$$\begin{array}{c} S_1 \diagdown S_2 \\ \end{array} \quad \begin{array}{cc} C & D \end{array}$$
$$\begin{array}{c} C \\ D \end{array} \begin{pmatrix} R/R & S/T \\ T/S & P/P \end{pmatrix}$$

It is plausible to assume $T > S, T > P$ and $R > \frac{S+T}{2}$. If $T > R > S > P$ the game is called Prisoner's Dilemma.

In the case of the Prisoner's Dilemma it is clear how a rational person has to behave: he has to defect. Many writers (e.g. [13]) have pointed out the dynamical character of social life. People meet each other again and again, and so they change their behaviour according to their expectations for the reactions of others. The fact that they usually don't know, when they will meet someone the last time, finds its mathematical analogue in the assumption of an infinite sequence of meetings. The payoffs, a player gains in

[2] Hobbes excludes the existence of a person, who is distinguished by nature in such a way, that he might be king by his own force.

such an iterated game, can be evaluated by considering an expression of the form

$$\mathbf{D} := \lim_{N\to\infty} \sum_{n=1}^{N} w_n \, A_n,$$

where A_n means the payoff a player gets in the n-th game, and w_n is a weight. In the literature one can find $w_n = \frac{1}{N}$ and $w_n = g^n$ for $g \in]0,1[$ as such weights. We define

$$\bar{D} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} A_n \qquad \text{and} \qquad D_g := \sum_{n=1}^{\infty} g^{n-1} \, A_n,$$

(g makes events in the far future less important).

A strategy of a player consists in the probability of C in the first game and a function which assigns a probability to C in the $(n+1)$-th game depending on the outcomes of the first n games.

One could now try to find a satisfying answer to the question, whether people are able to receive social cooperation on their own, in the following way:

First figure out, which strategy is best in the game that models the state-of-nature-situations. This strategy would be best for all participants, therefore they would use it. Then find out, whether two players using this strategy will be cooperating or defecting on the long run.

Unfortunately this attempt of solving the problem fails in several ways. Axelrod proved, that there is no best strategy in iterated Prisoner's Dilemma games (PD-games), where 'best strategy' means optimal strategy in the sense that an optimal strategy $\mathcal{S}$ gets against every other strategy $\mathcal{S}'$ a total payoff D_g, which cannot be improved by any other strategy playing against $\mathcal{S}'$. In the first section of this paper I will prove, that there is no optimal strategy with respect to $\bar{D}$ either.

The negative result of Axelrod has divided the successing research work into two groups:

1.) rational behaviour is defined by estimation of expectations (e.g. [4], [5], [8])

2.) computersimulations decide whether one considers a strategy as a good strategy (see [17]).

I will not refer to group one in my paper. In the computer tournaments performed by Axelrod, the strategy TIT FOR TAT has been very effective. Since then TIT FOR TAT has become omnipresent in the relevant philosophical literature.

Though TIT FOR TAT does have some remarkable advantages, for instance concerning the question of collective stability ([1], [17]), the question how successful in any suitable sense TIT FOR TAT really is, has not been studied yet. My paper is concerned with the question of the quality of TIT FOR TAT and, more general with alternative candidates for good strategies in (2×2)-matrix-games. Let me add some definitions and notations before starting my considerations:

A memory-one-strategy is represented by the probability of C in the first game and the vector (p_R, p_S, p_T, p_P), where p_i ($i \in \{R, S, T, P\}$) is the probability for C, supposed the outcome in the preceeding game was i. This vector will be denoted the vector of the probabilities of cooperation. Let $q_n(i)$ be the probability that a player gains i, ($i \in \{R, S, T, P\}$) in the n-th game, then there is a (4×4)-matrix U such that

$$(q_{n+1}(R), q_{n+1}(S), q_{n+1}(T), q_{n+1}(P))^{tr} = U \cdot (q_n(R), q_n(S), q_n(T), q_n(P))^{tr}.$$

We call U the transition-matrix.

Further, we set $D_N := \frac{1}{N}\sum_{n=1}^{N} A_n$. For a given memory-one-strategy $\mathcal{M}$, which against a strategy with $\mathbf{p}$ as vector of the probabilities of cooperation gains the average payoff $\bar{D}(\mathbf{p})$, we calculate the integral-mean $\tilde{D}$ of the average payoffs by

$$\tilde{D} = \int_{[0,1]^4} \bar{D}(\mathbf{p})\ d\mathbf{p}.$$

We will draw attention to the following strategies

TIT FOR TAT	....	$(1,0,1,0)$,	C as first move
PAVLOV	....	$(1,0,0,1)$	

As already announced the following statement holds:

Prop. 0.1 *In iterated PD-games there is no optimal strategy $\mathcal{S}$ in the sense, that $\mathcal{S}$ gains against every strategy $\mathcal{S}'$ an average payoff $\bar{D}$, which cannot be improved by any other strategy playing against $\mathcal{S}'$.*

To see this you only have to consider the strategy one could call ETERNAL PREJUDICE and its negation. ETERNAL PREJUDICE is the strategy, that takes over the behavior of the opponent in the first move for all the following games (the first move of ETERNAL PREJUDICE is irrelevant here). When playing against ETERNAL PREJUDICE one has to cooperate in the first move to achieve an optimal payoff. When playing against its negation one has to defect. Therefore ETERNAL PREJUDICE and its negation have different optimal opponents.

1 TIT FOR TAT

Is TIT FOR TAT a successful strategy?

As said in the previous paragraph there is no optimal strategy in the sense of universally optimizing $\tilde{D}$. But because of continuity considerations, there must be a strategy, which maximizes $\tilde{D}$ in the integral-mean under all memory-one-strategies, assuming uniform distribution. This strategy will turn out *not* to be TIT FOR TAT. So TIT FOR TAT is not a best strategy in the weaker sense either.

Theorem 1.1 *There is a strategy that is best in the integral-mean among all memory-one-strategies; this strategy is not TIT FOR TAT.*

At least in one sense TIT FOR TAT is not a bad strategy: Except very rare examples opponents of TIT FOR TAT don't gain more than TIT FOR TAT (but they don't gain less either). So if a player isn't interested in what he gains, but in his opponent's not gaining more than himself, that means, he is interested in not being the loser of the game, TIT FOR TAT is a good choice. For seeing this we need the following lemma, which can be found in the literature (see, e.g. [18]) and is a simple consequence of the lemma formulated in the Appendix.

Lemma 1.2 *Let $\bar{D}$ be the average payoff (in the sense of the preceding section), which TIT FOR TAT gains against a memory-one-strategy. Then for almost all memory-one-strategies $\mathcal{M}$ it holds*

$$\bar{D} = \lim_{N\to\infty} D_N = n(p_R, p_S, p_T, p_P) \cdot (R \cdot \frac{p_S}{1-p_R} + S + T + P \cdot \frac{1-p_T}{p_P})$$

with

$$n(p_P, p_S, p_T, p_P) = \frac{p_P(1-p_R)}{p_P p_S + 2p_P(1-p_R) + (1-p_T)(1-p_R)}$$

When TIT FOR TAT gets in a game the payoff $R \cdot q_R + S \cdot q_S + T \cdot q_T + P \cdot q_P$ then $\mathcal{M}$ gets $R \cdot q_R + T \cdot q_S + S \cdot q_T + Pq_P$. But from Prop. 1.2 follows that $\tilde{q}_S = \tilde{q}_T$; this yields,

Prop. 1.3 *For almost all memory-one-stragies $\mathcal{M}$ it holds: When playing against each other TIT FOR TAT and $\mathcal{M}$ gain the same (in the mean).*

It is possible to say explicitly how much TIT FOR TAT gains in the integral-mean against all memory-one-strategies:

Theorem 1.4. *TIT FOR TAT gains in the integral-mean over all memory-one-strategies the payoff*

$$\tilde{D} = R \cdot \tilde{q}_R + S \cdot \tilde{q}_S + T \cdot \tilde{q}_T + P \cdot \tilde{q}_P$$

where

$$\begin{aligned} \tilde{q}_S = \tilde{q}_T &= \tfrac{7}{2}\log 2 - \tfrac{9}{4}\log 3 + \tfrac{1}{96}(12 + \pi^2 + 6\log^2 2 + 24 \log 2 \log 3 + \\ &\quad + 12\mathrm{Li}_2(-\tfrac{1}{2}) - 6\mathrm{Li}_2(-\tfrac{1}{8})) = 0.21875101..., \\ \tilde{q}_R = \tilde{q}_P &= (1 - 2\tilde{q}_S)/2 = 0.28124899..., \end{aligned}$$

and $Li_2(.)$ is the dilogarithm, i.e. $\mathrm{Li}_2(z) = \int_z^0 \frac{\log(1-t)}{t}\, dt$ *or as a series expansion* $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$.

Is TIT FOR TAT a cooperative strategy?

Let us say 'Two strategies tend to mutual cooperation almost surely' if and only if $\tilde{q}_R = \lim_{n\to\infty} q_n(R) = 1$ and 'Two strategies tend rather to mutual cooperation than to a different outcome' if and only if $\tilde{q}_R = \lim_{n\to\infty} q_n(R) > \frac{1}{2}$.

Theorem 1.5 *TIT FOR TAT tends*

- *only with a nullset of memory-one-strategies almost surely to mutual cooperation.*
- *with only $9,51$ percent of the memory-one-strategies rather to mutual cooperation than to a different outcome.*
- *with all memory-one-strategies in the integral mean rather not to mutual cooperation (the probability is only $21,88$ percent).*

As TIT FOR TAT was successful in the computer tournaments performed by Axelrod one can easily have the idea to find a good strategy by improving TIT FOR TAT in some way. There have been several attempts to do so, one result was GENEROUS TIT FOR TAT or TIT FOR a TAT, which is not punishing *every* defection, but only every $\frac{1}{1-a}$-th one. As it turns out, this produces strategies, which of course are (by construction) more cooperative, but unfortunately less successful. A theorem analogous to Theorem 1.4 can be proved, the expressions to cope with become more complex, but remain of the same sort.

2 PAVLOV

Considering TIT FOR a TAT did not touch TIT FOR TAT's myth of being a good strategy. Just being more peaceful and otherwise sticking to the same strategy doesn't pay. PAVLOV is a strategy that makes its choice for a move according to agreement or disagreement in the previous game. It cooperates, if both players did the same in the previous game, otherwise it defects. For the proof of the following lemma see my remark concerning Lemma 1.2.

Lemma 2.1 *Let $\bar{D}$ be the average payoff, which PAVLOV gains against another strategy. Then for almost all memory-one-strategies $\mathcal{M}$ with (p_R, p_S, p_T, p_P) as the vector of the probabilities of cooperation it holds*

$$\bar{D} = \lim_{N\to\infty} D_N = n(p_R, p_S, p_T, p_P) \cdot (R \cdot \frac{p_P}{1-p_R} + S + T \cdot \frac{p_S}{1-p_T} + P).$$

$$\textit{with} \quad n(p_R, p_S, p_T, p_P) = \frac{(1-p_T)(1-p_R)}{p_P(1-p_T) + 2(1-p_R)(1-p_T) + p_S(1-p_R)}.$$

From this we immediately see that PAVLOV wins against almost all strategies, for which $p_S > 1-p_T$ holds, that means PAVLOV wins against half of all strategies, namely against those strategies, which prefer being exploited to exploiting. Against these strategies PAVLOV does better than TIT FOR TAT. Hence one could say, PAVLOV is not a 'friendly' strategy, as it practises on the opponent's weakness. Further and more important it holds

Theorem 2.2 *In the integral mean against all memory-one-strategies, PAVLOV gets a higher payoff than TIT FOR TAT.*

This can be easily seen by comparison of Lemma 1.2 and Lemma 2.1. Let us remark, that TIT FOR TAT and PAVLOV play draw against each other ($R, R, R, \ldots$ if PAVLOV cooperates in the first round; $S, P, T, S, \ldots$ if it defects).

3 Conclusion

Very large, pluralistic societies can be approximately modelled by a continuum of uniformly distributed strategies. There is (at least) one most powerful (successful) member in such an idealized society. This person might be seen as a natural head of the society. TIT FOR TAT has the reputation of being a good strategy, but it turns out, that it is definitely not the best strategy (Theorem 1.1). PAVLOV might be a/the best strategy, anyway it is more successful than TIT FOR TAT (Theorem 2.2). In spite of the fact, that TIT FOR TAT is not quite cooperative (Theorem 1.5), the superiority of PAVLOV is not a pleasing perspective, either. This becomes evident, if you compare some other properties of TIT FOR TAT and PAVLOV: Having successfully exploited the opponent PAVLOV tries it once again - unlike TIT FOR TAT. And also unlike TIT FOR TAT PAVLOV rewards mutual defection by cooperation as its next move.

The rational society does not exist, because the rational individual does not exist. The counterpart to the rational society, and perhaps the only remaining alternative is the pluralistic society. In the pluralistic society there is no one, who is distinguished by her/his rationality, but there is nevertheless someone, who is distinguished by her/his success. Assuming that such a person will be influential, it is of great interest to know, what this person will behave like. She/he will not choose TIT FOR TAT; it remains an open question, whether she/he will choose PAVLOV. In my paper I assumed a totally pluralistic society, i.e., I chose uniform distribution of strategies. For future research it would certainly be a rewarding task to widen the range of distributions to be considered.

Appendix

Proof of Theorem 1.4

According to Lemma 1.2 we have to calculate

$$\tilde{D} = R \cdot \tilde{q}_R + S \cdot \tilde{q}_S + T \cdot \tilde{q}_T + P \cdot \tilde{q}_P$$

with

$$\tilde{q}_R = \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 n(p_R, p_S, p_T, p_P) \cdot \frac{p_S}{1 - p_R} \; dp_R \; dp_S \; dp_T \; dp_P$$

$$\tilde{q}_S = \tilde{q}_T = \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 n(p_R, p_S, p_T, p_P) \; dp_R \; dp_S \; dp_T \; dp_P,$$

$$\tilde{q}_P = \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 n(p_R, p_S, p_T, p_P) \cdot \frac{1 - p_T}{p_P} \; dp_R \; dp_S \; dp_T \; dp_P$$

where $n(p_R, p_S, p_T, p_P)$ is defined as in Prop. 1.2.

Let us turn first to the integral

$$\tilde{q}_S = \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{p_P(1 - p_R)}{p_P p_S + 2p_P(1 - p_R) + (1 - p_T)(1 - p_R)} \; dp_R \; dp_S \; dp_T \; dp_P.$$

After substituting $p_T = r \sin\phi$, $1 - p_R = r \cos\phi$, $1 - p_T = s \sin\psi$, $p_P = s \cos\psi$ one gets,

$$\tilde{q}_S = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\bar{s}(\psi)}\int_0^{\bar{r}(\phi)} \frac{rs}{2+\tan\phi+\tan\psi}\; dr\; ds\; d\phi\; d\psi,$$

with $\bar{r}(\phi) = \begin{cases} 1/\cos\phi & \text{for } \phi < \pi/4 \\ 1/\sin\phi & \text{for } \phi > \pi/4 \end{cases}$ and $\bar{s}(\psi) = \begin{cases} 1/\cos\psi & \text{for } \psi < \pi/4 \\ 1/\sin\psi & \text{for } \psi > \pi/4 \end{cases}$.

Further

$$\tilde{q}_S = I_{S1} + I_{S2} + I_{S3} + I_{S4}$$

with

$$I_{S1} = \frac{1}{4}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos^2\phi}\frac{1}{\cos^2\psi}}{2+\tan\phi+\tan\psi}\; d\phi\; d\psi = \frac{1}{4}\int_0^1\int_0^1 \frac{1}{2+x+y}\; dx\; dy =$$
$$= \frac{5}{2}\log 2 - \frac{3}{2}\log 3 = 0.08494951...$$

$$I_{S2} = \frac{1}{4}\int_0^{\frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\frac{1}{\sin^2\phi}\frac{1}{\cos^2\psi}}{2+\tan\phi+\tan\psi}\; d\phi\; d\psi = \frac{1}{4}\int_0^1\int_0^1 \frac{1}{2+\frac{1}{x}+y}\; dx\; dy$$
$$= \frac{2}{3}\log 2 - \frac{3}{8}\log 3 = 0.05011851...$$

$I_{S3} = I_{S2}$ because of symmetry arguments

$$I_{S4} = \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\frac{1}{\sin^2\phi}\frac{1}{\sin^2\psi}}{2+\tan\phi+\tan\psi}\; d\phi\; d\psi = \frac{1}{4}\int_0^1\int_0^1 \frac{1}{2+\frac{1}{x}+\frac{1}{y}}\; dx\; dy$$
$$= \frac{1}{4}\int_0^1 \frac{2-\log(3-\frac{1}{y})+\frac{1}{y}}{(2+\frac{1}{y})^2}\; dy =$$
$$= \frac{1}{96}(12+\pi^2+6\log^2 2 - 32\log 2 + 24\log 2\log 3 +$$
$$+ 12\,\mathrm{Li}_2(-\tfrac{1}{2}) - 6\,\mathrm{Li}_2(-\tfrac{1}{8})) = 0.03356448....$$

As besides $\tilde{q}_S = \tilde{q}_T$ it holds that $\tilde{q}_R = \tilde{q}_P$, one can calculate $\tilde{q}_R = \tilde{q}_P = \frac{1-2\tilde{q}_S}{2}$.

Proof of Theorem 1.5

The first statement follows from Lemma 1.2.

From Theorem 1.4 the third statement can be taken directly: $\tilde{q}_R = 0.2187...$.

To derive the second statement we have to determine the volume of that part of the 4-dimensional unite cube, which satisfies $\bar{q}_R > \frac{1}{2}$, that is $p_P p_S > (1-p_R)(2p_P+(1-p_T))$. Therefore we have to consider

$$1 - V = \iiiint_B dp_R\; dp_S\; dp_T\; dp_P,$$

where B is $\{(p_R, p_S, p_T, p_P) \in [0,1]^4 : \frac{p_S}{1-p_R} < 2 + \frac{1-p_T}{p_P}\}$.

After suitable substitutions we get

$$1 - V = \frac{3}{4}\int_0^1\int_0^1 dx\; dy + \frac{1}{4}\iint_{\substack{\{(x,y)\in[0,1]^2:\\ x>y/(2y+1)\}}} dx\; dy = \frac{7}{8} + \frac{1}{16}\log 3 = 1 - 0.09517992...\ .$$

For the proof of Theorem 1.1, Lemma 1.2 and Lemma 2.1 one needs the following lemma, which can be proved by some largescaled, but routine considerations from linear algebra.

Lemma[3]: *Let $\mathcal{M}$ be a memory-one-strategy. The average payoff, that $\mathcal{M}$ gains against a memory-one-strategy $\mathcal{M}'$, we call $\bar{D} = \bar{D}(\mathcal{M}, \mathcal{M}')$.*

Further, let $U = U(\mathcal{M}, \mathcal{M}')$ be the transition matrix, that is the (4×4)-matrix, which fulfills $\mathbf{q}_n = U\mathbf{q}_{n-1}$ $(n \in \mathbb{N})$.

Then for almost all memory-one-strategies $\mathcal{M}'$, that means all with the exception of a 4-dimensional nullset, it holds

$$\bar{D} = \lim_{N\to\infty} D_N = \langle \mathbf{e}, \mathbf{a} \rangle,$$

where $\mathbf{a}$ is the payoff-vector (R, S, T, P) and $\mathbf{e}$ is the eigenvector, normed by $\| \cdot \|_1$, to the eigenvalue 1 of the matrix U.

Notice, that we are concerned with an only 4-dimensional nullset, which is important for the following proof. In the literature the analogous lemma is stated for $(4+4)$-dimensional nullsets of pairs of strategies.

Proof of Theorem 1.1

We have to prove that there is a strategy for which the integral mean $\tilde{D}$ attains its maximum.

The function

$$\tilde{D}(\mathbf{p}) := \int_{[0,1]^4} \bar{D}(\mathbf{p},\ \mathbf{p}')\ d\mathbf{p}'$$

with $\mathbf{p} = (p_1, ..., p_{4^d})$, $\quad \mathbf{p}' = (p'_1, ..., p'_{4^d})$

is well-defined, as $\tilde{D}$ is continous except on a nullset. We will now proof that $\tilde{D}(\mathbf{p})$ is continous on the whole cube $[0,1]^4$

For $\mathbf{p}, \mathbf{p}_0 \in [0,1]^4$ it holds

$$|\tilde{D}(\mathbf{p}) - \tilde{D}(\mathbf{p}_0)| \leq \int_{[0,1]^4} |\bar{D}(\mathbf{p}, \mathbf{p}') - \bar{D}(\mathbf{p}_0, \mathbf{p}')| d\mathbf{p}'.$$

We want to show, that the integrand is arbitrarily small for sufficiently near points $\mathbf{p}$ and $\mathbf{p}_0$, if $\mathbf{p}'$ doesn't lie in a certain nullset. For this purpose we assume to the contrary that

$$\lim_{\mathbf{p}\to\mathbf{p}_0} \bar{D}(\mathbf{p}, \mathbf{p}') \neq \bar{D}(\mathbf{p}_0, \mathbf{p}')$$

would hold for a set of vectors $\mathbf{p}$, which is not a nullset. ($\mathbf{p}_0$ is a constant not depending on $\mathbf{p}'$.) Then $U(\mathbf{p}_0, \mathbf{p}')$ would have a second Jordan-basis-vector with norm 1 to the eigenvalue 1, which is a contradiction to the preceeding lemma.

[3]This lemma can be extended to memory-d-strategies, but the dimension of the nullset has to be enlarged in a proper way.

Therefore for an arbitrary ε we can choose δ in such a way that from $|\mathbf{p} - \mathbf{p}_0| < \delta$ for almost all $\mathbf{p}'$ it follows

$$|\bar{D}(\mathbf{p}, \mathbf{p}') - \bar{D}(\mathbf{p}_0, \mathbf{p}')| < \varepsilon.$$

We get

$$|\tilde{D}(\mathbf{p}) - \tilde{D}(\mathbf{p}_0)| \leq \int_{[0,1]^4} \varepsilon \; d\mathbf{p}' = \varepsilon.$$

Hence $\bar{D}$ is continuous and attains its maximum on the compact unit cube.

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